FIG. 1A

Transfer between states

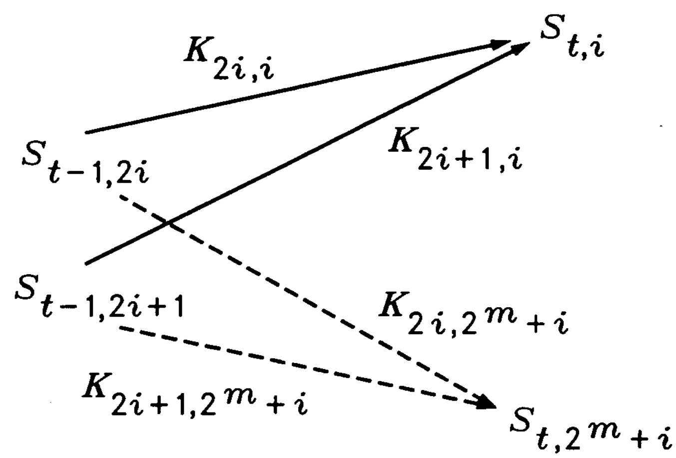

FIG. 1B

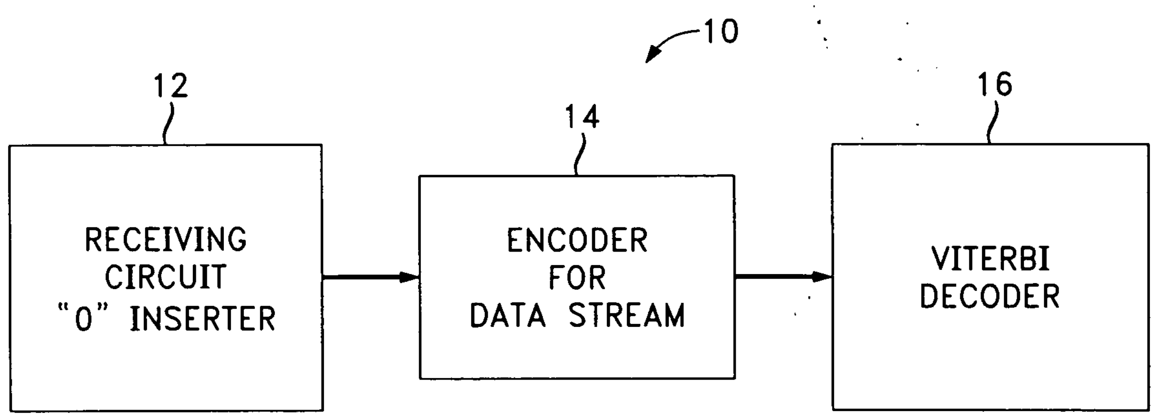

FIG. 2

$$s_{t,i} = D^{k\,2i,i} s_{t-1,2i} + D^{k\,2i+1,i} s_{t-1,2i+1},$$

$$s_{t,2^m+i} = D^{k\,2i,2^m+i} s_{t-1,2i} + D^{k\,2i+1,2^m+i} s_{t-1,2i+1}.$$

Thus, let $a_{2^{m-1}} = D^{k\,1,2^{m-1}}$ but

$$a_j = \left[D^{k\,2j,j}, D^{k\,2j+1,j}\right],\ j \neq 2^{m-1},$$

as a result:

$$S_t = \begin{bmatrix} S_{t,1} \\ \vdots \\ S_{t,2^m-1} \end{bmatrix} = TS_{t-1}, \qquad S_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D^{k\,0,2^{m-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where

$$T = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & a_{2^{m-1}-1} \\ a_{2^{m-1}} & 0 & 0 & \cdots & 0 \\ 0 & a_{2^{m-1}+1} & 0 & \cdots & 0 \\ 0 & 0 & a_{2^{m-1}+2} & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & a_{2^m-1} \end{bmatrix}.$$

Now, let

$$\vec{\xi} = \sum_{t=0}^{\infty} S_t, \tag{1}$$

then

$$\vec{\xi} = \sum_{t=0}^{\infty} T^t S_0 = (I-T)^{-1} S_0.$$

FIG. 3

$$s_{t,0} = D^{k}{}_{1,0}\, s_{t-1,1}, \text{let}$$

$$\xi_0 = \sum_{t=0}^{\infty} s_{t,0} = \left[\; D^{k}{}_{1,0} \quad 0 \quad \ldots \quad 0 \;\right] \vec{\xi}$$

$$= \left[\; D^{k}{}_{1,0} \quad 0 \quad \ldots \quad 0 \;\right] (I - T)^{-1} S_0. \quad (2)$$

FIG. 4

$$T_0 = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_{2^{m-1}-1} \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

we have

$$S_t = T_0 S_{t-1}.$$

Assume that the zero is inserted at the $j_0 - th$ position after the error one happens. Then

$$S_{nK+j} = \begin{cases} T^j S_{nK}, & 0 \leq j < j_0 \\ T^{j-j_0} T_0 T^{j_0-1} S_{nK}, & j_0 \leq j < K. \end{cases}$$

Now, let

$$P = I + \ldots + T^{j_0-1} + T_0 T^{j_0-1} + \ldots + T^{K-j_0} T_0 T^{j_0-1},$$

$$T_K = T^{K-j_0} T_0 T^{j_0-1},$$

we have

$$S_{nK} = T_K S_{(n-1)K},$$

$$\vec{\xi} = \sum_{n=0}^{\infty} \sum_{j=0}^{K-1} S_{nK+j} = \sum_{n=0}^{\infty} PS_{nK}$$

$$= P \sum_{n=0}^{\infty} T_K^n S_0 = P\,(I - T_K)^{-1} S_0$$

FIG. 5

$$T = \begin{bmatrix} 0 & D & D \\ 1 & 0 & 0 \\ 0 & D & D \end{bmatrix}.$$

as a result:

$$[D^2, 0, 0]\ (I - T)^{-1} \begin{bmatrix} 0 \\ D^2 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{vmatrix} 0 & -D & -D \\ 1 & 1 & 0 \\ 0 & -D & 1-D \end{vmatrix}}{\begin{vmatrix} 1 & -D & -D \\ -1 & 1 & 0 \\ 0 & -D & 1-D \end{vmatrix}} D^4$$

$$= \frac{D^5}{1-2D}.$$

FIG. 6

$$P = \begin{bmatrix} 1 & D & D \\ 0 & 1+D & D \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_K = \begin{bmatrix} 0 & D^2 & D^2 \\ 0 & 0 & 0 \\ 0 & D^2 & D^2 \end{bmatrix},$$

$$\xi_0 = [1, D, D] \begin{bmatrix} 1 & -D^2 & -D^2 \\ 0 & 1 & 0 \\ 0 & -D^2 & 1-D^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} D^4$$

$$= \frac{\begin{vmatrix} 1 & -D^2 & -D^2 \\ 1 & D & D \\ 0 & -D^2 & 1-D^2 \end{vmatrix}}{\begin{vmatrix} 1 & -D^2 & -D^2 \\ 0 & 1 & 0 \\ 0 & -D^2 & 1-D^2 \end{vmatrix}} D^4$$

$$= \frac{D^5}{1-D} = D^5 \sum_{k=0}^{\infty} D^k.$$

FIG. 7

$$P = \begin{bmatrix} 1+D & D+D^2 & D+D^2 \\ 1 & 1 & 0 \\ 0 & D & 1+D \end{bmatrix},$$

$$T_K = \begin{bmatrix} 0 & 0 & 0 \\ D & D^2 & D^2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \xi_0 &= [1,D,D] \times \\ &\quad \begin{bmatrix} 1 & 0 & 0 \\ -D & 1-D^2 & -D^2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} (1+D)D^4 \\ &= \frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & D & D \\ 0 & 0 & 1 \end{vmatrix} (1+D)D^4}{\begin{vmatrix} 1 & 0 & 0 \\ -D & 1-D^2 & -D^2 \\ 0 & 0 & 1 \end{vmatrix}} \\ &= \frac{D^5}{1-\mathrm{D}} = D^5 \sum_{k=0}^{\infty} D^k. \end{aligned}$$